

A geometric model for Hochschild homology of Soergel bimodules

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Abstract An important step in the calculation of the triply graded link homology theory of Khovanov and Rozansky is the determination of the Hochschild homology of Soergel bimodules for $SL(n)$. We present a geometric model for this Hochschild homology for any simple group G , as equivariant intersection homology of $B \times B$ -orbit closures in G . We show that, in type A these orbit closures are equivariantly formal for the conjugation T -action. We use this fact to show that in the case where the corresponding orbit closure is smooth, this Hochschild homology is an exterior algebra over a polynomial ring on generators whose degree is explicitly determined by the geometry of the orbit closure, and describe its Hilbert series, proving a conjecture of Jacob Rasmussen.

1 Introduction

In this paper, we consider the Hochschild homology of Soergel bimodules, and construct a geometric interpretation of it. This will allow us to explicitly compute the Hochschild homology of a special class of Soergel bimodules.

Soergel bimodules are bimodules over a polynomial ring, which appear naturally both in the study of perverse sheaves on flag varieties and of the semiring of projective functors on the BGG category \mathcal{O} . Recently interest in them has been rekindled by the appearance of connections with link homology as shown by Khovanov [Kh].

Khovanov's work showed that one aspect of Soergel bimodules which had not been carefully studied up to that date was, in fact, of great importance: their Hochschild homology. While the operation of taking Hochschild homology is

hard to motivate from a representation theoretic perspective, we argue that it is, in fact, naturally geometric, rather than purely combinatorial/algebraic.

Let G be a connected reductive complex algebraic group, with Lie algebra \mathfrak{g} . Let B be a Borel of G , T a Cartan subgroup of G , \mathfrak{t} be its Lie algebra, $n = \dim T$ be the rank of G , and $W = N_G(T)/T$ be the Weyl group of G . For any $w \in W$, we let $G_w = \overline{BwB}$. Note that T is a deformation retract of B , so, we have $H_B^*(X) \cong H_T^*(X)$ for all B -spaces X . We will freely switch between B - and T -equivariant cohomology throughout this paper.

We note that G_w is a closed subvariety of G which is smooth if and only if the corresponding Schubert variety $\mathcal{B}_w \subset G/B = \mathcal{B}$ is (and thus G_w typically singular). Note that $B \times B$ acts on G_w by left and right multiplication, and that restricting to the diagonal, we get the action of B by conjugation. Of course, we also have left and right actions of B but we will never consider these separately. We will **always** mean the conjugation action.

Now consider the graded ring $S = H_T^*(pt, \mathbb{C}) = \mathbb{C}[t^*]$, which is endowed with the obvious W -action. Given a simple reflection s , denote by R_s the bimodule $S \otimes_{S^s} S[1]$ where S^s is the subring of invariants under the reflection s and, by convention, $[a]$ denotes the grading shift $(M[a])^i = M^{i+a}$.

We now come to the definition of Soergel bimodules:

Definition 1 A **Soergel bimodule** is (up to grading shift) a direct summand of the tensor product $R_{\mathbf{i}} = R_s \otimes_S R_t \otimes_S \cdots \otimes_S R_u$ in the category of graded S -bimodules, where $\mathbf{i} = (s, t, \dots, u)$ is a sequence of (not necessarily distinct) simple reflections.

The motivation for studying Soergel bimodules comes from representation theory, geometry and connections between the two. In fact, one has the following theorem:

Theorem 1.1 (Soergel) *The indecomposable Soergel bimodules are parameterized (up to a shift in the grading) by the Weyl group [So4, Satz 6.14].*

The irreducible Soergel bimodule corresponding to $w \in W$ may be obtained as $R_w \cong IH_{B \times B}^(G_w)$ [So3, Lemma 5], where IH^* denotes intersection cohomology as defined by Goresky and MacPherson [GM].*

We choose our grading conventions on IH^* so that the 0th degree is the mirror of Poincaré duality, that is, so that the lowest degree component of R_w sits in degree $-\ell(w)$.

More generally, Soergel bimodules can be identified with the $B \times B$ -equivariant hypercohomology $\mathbb{H}_{B \times B}^*(G, \mathcal{F})$ of sums \mathcal{F} of shifts of $B \times B$ -equivariant semi-simple perverse sheaves on G .

Our central result is a geometric description of Hochschild homology $HH_*(-)$ of these modules.

Theorem 1.2 *Let \mathcal{F} be a semi-simple $B \times B$ -equivariant perverse sheaf on G and let R be its $B \times B$ -equivariant cohomology. Then*

$$HH_*(R) \cong \mathbb{H}_B^*(G, \mathcal{F}),$$

where B acts by conjugation (i.e., by the diagonal inclusion $B \hookrightarrow B \times B$). In particular, we have

$$HH_*(R_w) \cong IH_B^*(G_w).$$

Unfortunately, $IH_B^*(G_w)$ has a single grading, whereas $HH_*(R_w)$ has two: by decomposition into the components HH_i (“the Hochschild grading”), and one coming from the grading on R_w (“the polynomial grading”). This isomorphism takes the single grading on $IH_B^*(G_w)$ to the difference of the two gradings on $HH_*(R_w)$.

We can give a geometric interpretation of these gradings, but in a somewhat roundabout manner. A result of Rasmussen [Ra] shows that

Theorem 1.3 *Assume $G = \mathrm{SL}(n)$ or $\mathrm{GL}(n)$. Then $\mathbf{IC}(G_w)$ is equivariantly formal. Thus, the map*

$$\iota_T^* : IH_T^*(G_w) \rightarrow \mathbb{H}_T^*(T, \mathbf{IC}(G_w)) \cong H_T^*(T) \otimes \mathbf{IC}(G_w)_e$$

is injective and an isomorphism after tensoring with the fraction field of S .

Furthermore, since $H_T^*(T, \mathbf{IC}(G_w)) \cong S \otimes_{\mathbb{C}} H^*(T, \mathbf{IC}(G_w))$ by the Künneth theorem, we can equip this S module with a bigrading, with an “equivariant” grading from the first factor, and a “topological” grading from the second.

Theorem 1.4 *The intersection cohomology $IH_B^*(G_w)$ obtains “topological” and “equivariant” gradings by transport of structure from the map ι^* and $HH_*(R_w)$ obtains the same by the isomorphism of Theorem 1.2. These are related to the “Hochschild” and “polynomial” gradings by*

$$\deg_t(\gamma) = \deg_h(\gamma) \quad \deg_e(\gamma) = \deg_p(\gamma) - 2 \deg_h(\gamma)$$

where $\deg_*(x)$ denotes the degree of x in the grading whose name begins with the letter $*$.

The case where G_w is smooth is of special interest to us, so any reader who is unhappy with the presence of intersection cohomology and perverse sheaves can restrict to that case, in which case intersection cohomology is canonically isomorphic to Čech cohomology.

Theorem 1.5 *If G_w is smooth (in any type), then the Hochschild homology of R_w is of the form*

$$HH_*(R_w) = \wedge^\bullet(\gamma_1, \dots, \gamma_n) \otimes_{\mathbb{C}} S$$

where $\{\gamma_i\}_{i=1, \dots, m}$ are generators with

$$\deg_h(\gamma_i) = 1 \quad \deg_p(\gamma_i) = 2k_i$$

for positive integers k_i determined by the geometry of G_w .

These integers can be calculated using the action of w on the root system, or in $SL(n)$ by presenting G_w/B as an iterated Grassmannian bundle.

While the indecomposable modules R_w are perhaps most natural from the perspective of geometry or representation theory, definition (1) (and the study of knot homology, which we discuss briefly in Section 2) encourages us to concentrate on the modules R_i . We call these particular Soergel bimodules **Bott-Samelson** for reasons which will be clarified in Section 5.

Bott-Samelson modules are naturally identified with the equivariant homology of the “groupy” Bott-Samelson space

$$G_i \cong P_s \times_B P_t \times_B \cdots \times_B P_u.$$

and essentially the analogues of all appropriate theorems connecting the $B \times B$ -orbit closures in G with Soergel bimodules are true here.

Theorem 1.6 *If $G = SL(n)$ or $GL(n)$, then for all i , we have*

$$R_i \cong H_{B \times B}^*(G_i) \quad HH_*(R_i) \cong H_B^*(G_i).$$

The T -conjugation on G_i is equivariantly formal, and the injection

$$i_T^* : H_T^*(G_i) \rightarrow H_T^*(G_i^T)$$

induces a bigrading on $H_T^(G_i)$ matching that on $HH_*(R_i)$ as in Theorem 1.4.*

The structure of the paper is as follows: In Section 2, we discuss the importance of the Hochschild homology of Soergel bimodules in knot theory. In Section 4, we prove Theorem 1.2, using the formalism of dg-modules, the relevant points of which we will summarize in Section 3. In Section 5, we will cover in more detail how to construct Soergel bimodules as equivariant intersection cohomology of various varieties. Finally, in Section 6, we prove Theorems 1.3–1.6.

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2 Knot homology and Soergel bimodules

While Soergel bimodules merit study simply by being connected so much delightful mathematics, we have applications in knot theory in mind, as we will now briefly describe. The interested reader can find more details in the papers of Khovanov [Kh], Rasmussen [Ra], and Webster [We].

The braid group B_G of G is a finitely presented group, with generators σ_s for each simple reflection $s \in W$, which is defined by the presentation

$$\begin{aligned}\sigma_s \sigma_t &= \sigma_t \sigma_s && (\text{when } (st)^2 = e) \\ \sigma_s \sigma_t \sigma_s &= \sigma_t \sigma_s \sigma_t && (\text{when } (st)^3 = e) \\ (\sigma_s \sigma_t)^2 &= (\sigma_t \sigma_s)^2 && (\text{when } (st)^4 = e) \\ (\sigma_s \sigma_t)^3 &= (\sigma_t \sigma_s)^3 && (\text{when } (st)^6 = e)\end{aligned}$$

Note that if $G = \mathrm{SL}(n)$, then $B_n = B_G$ is the standard braid group familiar from knot theory.

There are several natural weak actions of the braid group on category \mathcal{O} by families of functors (see, for example, [AS, KM]), which have an avatar on the

bimodule side of the picture in the form of a complex of bimodules attached to each braid group element. The description of these bimodule complexes can be found in various sources, for example [Kh], or for general Coxeter groups in [Ro].

Define the complexes of S -bimodules:

$$\begin{aligned} F(\sigma_s) &= \cdots \longrightarrow S[-1] \longrightarrow R_s \longrightarrow 0 \longrightarrow \cdots \\ F(\sigma_s^{-1}) &= \cdots \longrightarrow 0 \longrightarrow R_s \longrightarrow S[1] \longrightarrow \cdots \end{aligned}$$

where the maps between non-zero spaces are the unique (up to scalar) non-zero maps of degree 0. These maps are defined by (respectively) the push-forward and pullback in $B \times B$ -equivariant cohomology for the inclusion $B \hookrightarrow \overline{BsB}$.

Theorem 2.1 *The shuffling complex*

$$F(\sigma) = F(\sigma_{i_1}^{\epsilon_1}) \otimes_S \cdots \otimes_S F(\sigma_{i_m}^{\epsilon_m})$$

of a braid $\sigma = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_m}^{\epsilon_m}$ (where $\epsilon_i = \pm 1$) depends up to homotopy only on σ not its factorization. In particular,

$$F(\sigma\sigma') = F(\sigma) \otimes_S F(\sigma'),$$

so F defines a categorification of B_G .

The maps in this complex also have a geometric interpretation: each degree is a direct sum of Bott-Samelson modules for subsequences of \mathbf{i} , and the “matrix coefficients” of the differential between these are induced by pullback or push-forward maps on $B \times B$ -equivariant cohomology for inclusions $G_{\mathbf{i}'} \rightarrow G_{\mathbf{i}''}$ of Bott-Samelson spaces where \mathbf{i}' and \mathbf{i}'' are subsequences of \mathbf{i} which differ by a single index.

Even better, this complex can be used to find a knot invariant, as was shown by Khovanov [Kh]. Let $HH_*(R)$ be the Hochschild homology of R , which can be defined (using the standard equivalence between $S - S$ -bimodules with $S \otimes S^{op}$) by

$$HH_i(R) = \mathrm{Tor}_{S \otimes S^{op}}^i(S, R).$$

This can be calculated by the Hochschild complex of S (which is often used as a definition), or by the Koszul complex, both of which are free resolutions of S as an $S \otimes S^{op}$ -module.

In the case where $G = \mathrm{SL}(n)$, Hochschild homology is a categorification of the trace on the braid group defined by Jones [Jo]. Remarkably, combining these creates a categorification of knot polynomials, which had previously been defined by Khovanov and Rozansky.

Theorem 2.2 (Khovanov, [Kh]) *As a graded vector space, the homology $\mathcal{KR}(\bar{\sigma})$ of the complex $HH_i(F(\sigma))$ depends only on the knot $\bar{\sigma}$, and in fact, is precisely the triply graded homology defined by Khovanov and Rozansky in [KR].*

Applying Theorem 1.2 to our remarks above, we can understand the differentials of the complex $HH_i(F(\sigma))$ in terms of pullback and pushforward on B -equivariant cohomology.

3 The equivariant derived category and dg-modules

Since our readers may be less well-acquainted with the formalism of equivariant derived categories and their connection with dg-modules, as developed by Bernstein and Lunts, in this section, we will provide a brief overview of the necessary background for later sections. This material is discussed in considerably more detail in their monograph [BL].

Suppose a Lie group G operates on a space X . We have maps:

$$\begin{aligned} m: G \times X &\rightarrow X & m(g, x) &= g \cdot x \\ \pi: G \times X &\rightarrow X & \pi(g, x) &= x \end{aligned}$$

A function f on X is G -invariant if and only if $m^*f = \pi^*f$. It is therefore natural to define a G -equivariant sheaf on X to be a sheaf \mathcal{F} on X together with an isomorphism $\theta: m^*\mathcal{F} \rightarrow \pi^*\mathcal{F}$. (There is also a cocycle condition that we ignore here).

One can show that if G operates topologically freely on X with quotient X/G then the categories of G -equivariant sheaves on X and sheaves on X/G are equivalent.

Faced with a G -space, one would wish to define an “equivariant derived category”. This should associate to a pair (G, X) a triangulated category $D_G^b(X)$ together with a “forgetting G -equivariance” functor $D_G^b(X) \rightarrow D^b(X)$. For any reasonable definition of $D_G^b(X)$, there should be an equivalence $D_G^b(X) \cong D^b(X/G)$ if G acts topologically freely as well as notions of pullback and pushforward for equivariant maps.

The trick is to notice that, at least homotopically, we may assume that the action is free: we “liberate” X (i.e. make it free) by replacing it with $X \times EG$ where EG is the total space of the universal G -bundle (i.e. any contractible

space on which G acts freely). The first projection $p : X \times EG \rightarrow X$ is a homotopy equivalence (because EG is contractible) and the diagonal operation of G on $X \times EG$ is free. Thus, we can consider the quotient map $q : X \times EG \rightarrow X \times_G EG$ as “liberation” of $X \rightarrow X/G$. The following definition then makes sense:

Definition 2 *The (bounded) equivariant derived category $D_G^b(X)$ is the full subcategory of $D^b(X \times_G EG)$ consisting of complexes $\mathcal{F} \in D^b(X \times_G EG)$ such that $q^*\mathcal{F} \cong p^*\mathcal{G}$ for some complex $\mathcal{G} \in D^b(X)$.*

In the case where X is a single point, we have $X \times_G EG \cong EG/G$, which is usually denoted BG .

Remark 1 This is not exactly Bernstein and Lunts’ definition. Consider the following diagram of spaces:

$$X \xleftarrow{p} X \times EG \xrightarrow{q} X \times_G EG$$

They define an equivariant sheaf to be a tuple $(\mathcal{G}, \mathcal{F}, \alpha)$ where $\mathcal{G} \in D^b(X)$, $\mathcal{F} \in D^b(X \times_G EG)$, and $\alpha : p^*\mathcal{G} \rightarrow q^*\mathcal{F}$ is an isomorphism. However, the functor to the above definition which forgets everything except for \mathcal{G} is an equivalence of categories.

Remark 2 As previously mentioned, it is natural to expect a “forgetting G -equivariance functor” $\text{For} : D_G^b(X) \rightarrow D^b(X)$. With p and q as in the previous remark, we may define $\text{For}(\mathcal{F}) = p_*q^*\mathcal{F}$. If $1 \hookrightarrow G$ is the inclusion of the trivial group, then $\text{For}(\mathcal{F}) \cong \text{res}_G^1\mathcal{F}$ (res_G^1 is defined below).

If $H \subset G$ is a subgroup, then we may take EG for EH and we have a natural map

$$\varphi_H^G : X \times_H EG \rightarrow X \times_G EG$$

commuting with the projection to X . The pullback and pushforward by this map induce functors

$$(\varphi_H^G)^* = \text{res}_H^G : D_G^b(X) \rightarrow D_H^b(X) \quad (\varphi_H^G)_* = \text{ind}_G^H : D_H^b(X) \rightarrow D_G^b(X)$$

Similarly, for any G -space we have a map $X \times_G EG \rightarrow BG$. If X is a reasonable space (for example a complex algebraic variety with the classical topology) push-forward yields a functor

$$\pi_* : D_G^b(X) \rightarrow D_G^b(pt)$$

which commutes with the induction and restriction functors.

In this work we are interested in equivariant cohomology for connected Lie groups. These emerge as the cohomology of objects living in $D_G(pt)$. The first key observation of Bernstein and Lunts' is the following:

Proposition 3.1 *If G is a connected Lie group then $D_G^b(pt)$ is the triangulated subcategory of $D^b(BG)$ generated by the constant sheaf.*

It turns out that this observation allows Bernstein and Lunts to give an algebraic description of $D_G(pt)$. For this we need the language of differential graded algebras and modules.

Definition 3 *A differential graded algebra (or dg-algebra) is a unital, graded associative algebra $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i$ together with an additive endomorphism $d : \mathcal{A} \rightarrow \mathcal{A}$ of degree 1 such that:*

- (1) *d is a differential: i.e. $d^2 = 0$.*
- (2) *d satisfies the Leibniz rule: $d(ab) = (da)b + (-1)^{\deg a} a(db)$.*
- (3) *$d(1_{\mathcal{A}}) = 0$, where $1_{\mathcal{A}}$ denotes the identity of \mathcal{A} .*

A left differential graded module (or left dg-module) over a differential graded algebra \mathcal{A} is a graded left \mathcal{A} -module M together with a differential $d_M : M \rightarrow M$ of degree 1 satisfying:

- (1) *$d_M^2 = 0$.*
- (2) *$d_M(am) = (da)m + (-1)^{\deg a} a(d_M m)$.*

A morphism of dg-modules is a graded \mathcal{A} -module homomorphism $f : M \rightarrow M'$ commuting with the differentials.

Remark 3 *If $\mathcal{A} = \mathcal{A}_0$ is concentrated in degree zero, then a differential graded module is just a chain complex of \mathcal{A} -modules.*

Given any dg-module \mathcal{M} we may consider $H^*(\mathcal{M})$, which is a graded module over the graded algebra $H^*(\mathcal{A})$. As with the category of modules over an algebra, the category of dg-modules over a dg-algebra has a homotopy category and a derived category as defined by Bernstein and Lunts [BL].

Definition 4 *A map of dg-modules $f : \mathcal{M} \rightarrow \mathcal{M}'$ is a **quasi-isomorphism** if the induced map $H^*(\mathcal{M}) \rightarrow H^*(\mathcal{M}')$ on cohomology is an isomorphism.*

The **derived category** of dg-modules for the dg-algebra \mathcal{A} , which we denote by $\mathrm{dgMod}\mathcal{A}$, is the category whose objects are dg-modules, and whose morphisms are compositions of chain maps and formal inverses to quasi-isomorphisms. We denote by $\mathrm{dgMod}^f\mathcal{A}$ the full subcategory consisting of dg-modules, finitely generated over \mathcal{A} .

Given a morphism $\mathcal{A} \rightarrow \mathcal{A}'$ of dg-algebras we would like to define functors of restriction and extension of scalars. Restriction of scalars is unproblematic (acyclic complexes are mapped to acyclic complexes) however a little more care is needed in defining extension of scalars. Just as in the normal derived category, one needs a special class of objects in order to define functors. In $\mathrm{dgMod}\mathcal{A}$ these are the \mathcal{K} -projective objects [BL] which we will not discuss in complete generality. In the sequel we will only be interested in a special class of dg-algebras in which it is possible to construct \mathcal{K} -projective objects rather explicitly:

Proposition 3.2 (Proposition 11.1.1 of [BL]) *Let $\mathcal{A} = \mathbb{C}[x_1, \dots, x_n]$ be viewed as a dg-algebra by setting $d_{\mathcal{A}} = 0$ and requiring that each x_i have even degree. Then all dg-modules which are free as \mathcal{A} -modules are \mathcal{K} -projective.*

Let us now describe how to construct a \mathcal{K} -projective resolution of a dg-module \mathcal{M} when \mathcal{A} is as in the proposition. Assume first that \mathcal{M} has zero differential. We may choose a free resolution in the category of graded \mathcal{A} -modules:

$$P_{-n} \rightarrow \cdots \rightarrow P_{-2} \rightarrow P_{-1} \rightarrow \mathcal{M}$$

We then consider the direct sum $\mathcal{P} = \bigoplus_i P_i[i]$ as a dg-module with the natural differential. For example, elements in P_{-2} are mapped under $d_{\mathcal{P}}$ into P_{-1} using the corresponding differential in the above resolution. The natural morphism $\mathcal{P} \rightarrow \mathcal{M}$ (again induced from the resolution above) is a quasi-isomorphism.

Using standard techniques from homological algebra, one may give a more elaborate construction of a \mathcal{K} -projective resolution for dg-modules over \mathcal{A} with non-trivial differential [BL], but this will not be necessary for our results.

We may now define the extension of scalars functor. Suppose we have a morphism $\mathcal{A} \rightarrow \mathcal{A}'$ of dg-algebras, and that \mathcal{A} and \mathcal{A}' are as in the proposition. For any $\mathcal{M} \in \mathrm{dgMod}\mathcal{A}'$ and $N \in \mathrm{dgMod}\mathcal{A}$ we define

$$\mathrm{ext}_{\mathcal{A}}^{\mathcal{A}'}(N) = \mathcal{A}' \overset{L}{\otimes}_{\mathcal{A}} N = \mathcal{A}' \otimes_{\mathcal{A}} P$$

where P is a \mathcal{K} -projective resolution of P . (Note that we may also take a \mathcal{K} -projective resolution of \mathcal{A}' as an \mathcal{A} dg-module).

We can now return to a discussion of the equivariant derived category. Abelian and triangulated categories can often be described by “module categories” over endomorphism rings of generators. We have already seen that $D_G^b(pt)$ is precisely the triangulated subcategory of $D^b(BG)$ generated by the constant sheaf. Bernstein and Lunts then consider the functor $\mathrm{Hom}_{D(BG)}(\mathbb{C}_{BG}, -)$ and notice that $\mathrm{End}(\mathbb{C}_{BG})$ has the structure of a dg-algebra. Moreover there is a quasi-isomorphism $\mathcal{A}_G = H^*(BG) \rightarrow \mathrm{End}(\mathbb{C}_{BG})$ of dg-algebras.

This yields a functor

$$\Gamma_G = \mathrm{Hom}_{D^b(BG)}(\mathbb{C}_{BG}, -) : D_G^b(pt) \rightarrow \mathrm{dgMod}^f \mathcal{A}_G.$$

Bernstein and Lunts then show:

Theorem 3.3 (Main Theorem of Bernstein-Lunts [BL]) *Assume as above that G is a connected Lie group. The above functor gives an equivalence commuting with the cohomology functor:*

$$\Gamma_G : D_G^b(pt) \rightarrow \mathrm{dgMod}^f \mathcal{A}_G$$

Moreover if $\varphi : G \rightarrow H$ is an inclusion of connected Lie groups and $\mathcal{A}_H \rightarrow \mathcal{A}_G$ is the induced homomorphism the restriction and induction functors have an algebraic description in terms of dg-modules:

$$\begin{array}{ccc} D_G^b(pt) & \xrightarrow{\mathrm{ind}_G^H} & D_H^b(pt) \\ \downarrow \Gamma_H & & \downarrow \Gamma_G \\ \mathrm{dgMod}^f \mathcal{A}_G & \xrightarrow{\mathrm{res. of sc.}} & \mathrm{dgMod}^f \mathcal{A}_H \end{array} \qquad \begin{array}{ccc} D_H^b(pt) & \xrightarrow{\mathrm{res}_H^G} & D_G^b(pt) \\ \downarrow \Gamma_H & \mathrm{ext}_{\mathcal{A}_H}^{\mathcal{A}_G} & \downarrow \Gamma_G \\ \mathrm{dgMod}^f \mathcal{A}_H & \xrightarrow{\quad} & \mathrm{dgMod}^f \mathcal{A}_G \end{array}$$

Remark 4 Note that, if G is a connected Lie group \mathcal{A}_G is always a polynomial ring on even generators, and hence, by the previous discussion, we can always construct a sufficient supply of finitely generated \mathcal{K} -projective objects.

We will now describe equivariant intersection cohomology complexes, which will be important in the sequel. Given a variety X (for simplicity assumed to be over the complex numbers), a smooth locally closed subvariety U , and a local system \mathcal{L} on U there is a complex $\mathrm{IC}(U) \in D^b(X)$ called the intersection cohomology complex extending \mathcal{L} , with remarkable properties (see for example [BBD] and [GM]).

It is possible to construct equivariant analogues of the intersection cohomology complexes, as described in Chapter 5 of [BL]: If X is furthermore a G -variety for a complex algebraic group G , U is a smooth G -stable subvariety, and \mathcal{L} is a G -equivariant local system on U then there exists an “equivariant intersection cohomology complex” which we will also denote $\mathbf{IC}(U, \mathcal{L})$. Forming intersection cohomology complexes behaves well with respect to restriction, as the following lemma shows:

Lemma 3.4 *If $H \hookrightarrow G$ is an inclusion of algebraic groups, X is a G -variety, U is a smooth G -stable subvariety then:*

$$\mathrm{res}_G^H \mathbf{IC}(U, \mathcal{L}) \cong \mathbf{IC}(U, \mathrm{res}_G^H \mathcal{L})$$

Remark 5 In dealing with equivariant intersection cohomology complexes it is more convenient to work with an equivalent definition of the equivariant derived category as a limit of categories associated to $X \times_G E_n G$, where $E_n G$ is an finite dimensional algebraic variety, which approximates EG ([BL]).

4 Hochschild homology and dg-algebras

Recall that the Hochschild homology of an $S \otimes S$ -module R can be defined as

$$HH_*(R) = S \overset{L}{\otimes}_{S \otimes S} R$$

where S has been made into an $S \otimes S$ algebra by the multiplication map. Since $S \cong \mathcal{A}_B$ and $S \otimes S \cong \mathcal{A}_{B \times B}$, this map is that induced by the diagonal group homomorphism $B \hookrightarrow B \times B$. Thus, we expect that the geometric analogue of taking Hochschild homology is restricting from a $B \times B$ -action to the diagonal B .

However, we must be careful about the difference between dg-modules and modules. Hochschild homology is an operation on $S - S$ -bimodules, not dg-bimodules. Thus, to make a precise statement requires us to restrict to formal complexes.

Definition 5 *Let $M \in \mathrm{dgMod} \mathcal{A}_G$ be a dg-module. If $M \cong H^*(M)$ in $\mathrm{dgMod} \mathcal{A}_G$ we say that M is **formal**. Similarly, $\mathcal{F} \in D_G(pt)$ is **formal** if $\Gamma_G(\mathcal{F})$ is.*

The following proposition connects the Hochschild cohomology of formal equivariant sheaves with another equivariant cohomology. This is our main technical tool.

Proposition 4.1 Suppose $\mathcal{F} \in D_{B \times B}(pt)$ is formal. Then one has an isomorphism:

$$\bigoplus_i HH_i(\mathbb{H}_{B \times B}^*(\mathcal{F}))[i] \cong \mathbb{H}_B^*(\text{res}_{B \times B}^B \mathcal{F})$$

as graded S -modules.

Furthermore this isomorphism is functorial. That is, if \mathcal{F} and \mathcal{G} are formal sheaves in $D_{B \times B}(pt)$, and $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism, the maps $\mathbb{H}_B^*(\varphi)$ and $HH_*(\mathbb{H}_{B \times B}^*(\varphi))$ commute with this isomorphism.

Proof of Proposition 4.1 In order to work out the Hochschild homology of $\mathbb{H}_{B \times B}^*(\mathcal{F})$ we may take a free resolution of $\mathbb{H}_{B \times B}^*(\mathcal{F})$ by $S \otimes S$ -modules:

$$0 \rightarrow P_{-2n} \rightarrow \cdots \rightarrow P_{-1} \rightarrow \mathbb{H}_{B \times B}^*(\mathcal{F})$$

We then apply $S \otimes_{S \otimes S} -$ and take cohomology. However, because $\mathbb{H}_{B \times B}^*(\mathcal{F}) \cong \Gamma_{B \times B}(\mathcal{F})$ in $\text{dgMod } \mathcal{A}_{B \times B}$, we may also regard $\bigoplus P_i[-i]$ as a \mathcal{K} -projective resolution of $\Gamma(\mathcal{F})$. By Theorem 3.3 we have:

$$\mathbb{H}_B^*(\text{res}_{B \times B}^B \mathcal{F}) \cong H^*(S \otimes_{S \otimes S}^L \mathbb{H}_{B \times B}^*(\mathcal{F})) \cong \bigoplus HH_i(\mathbb{H}_{B \times B}^*(\mathcal{F}))[i]$$

□

Proof of Theorem 1.2 Let \mathcal{F} denote the image of the intersection cohomology sheaf on \overline{BwB} in $D_{B \times B}(pt)$. We will see in the next section that \mathcal{F} is formal. Hence we can apply the above proposition. However, we also know that $\mathbb{H}_{B \times B}^*(\mathcal{F})$ is the indecomposable Soergel bimodule R_w . Hence:

$$\bigoplus_i HH_i(R_w)[i] \cong \mathbb{H}_B^*(\text{res}_{B \times B}^B \mathcal{F})$$

But $\text{res}_{B \times B}^B$ commutes with the map to a point and $\text{res}_{B \times B}^B(\mathbf{IC}(G_w)) \cong \mathbf{IC}(G_w)$ (Lemma 3.4). Hence:

$$\bigoplus_i HH_i(R_w)[i] \cong \mathbb{H}_B^*(\mathbf{IC}(G_w))$$

This then yields the main theorem.

□

5 The geometry of Bott-Samelson bimodules

In this section, we discuss Bott-Samelson bimodules, calculate their $B \times B$ -equivariant cohomology and obtain the formality results needed above.

Since we have already described the geometric realization of indecomposable bimodules, as intersection cohomology of subvarieties of G , we know abstractly that the Bott-Samelson bimodule $R_{\mathbf{i}}$ must be the hypercohomology of a perverse sheaf obtained by taking a direct sum of IC-sheaves of these subvarieties with appropriate multiplicities.

However, this is deeply dissatisfying from a geometric viewpoint, and totally at odds with our viewpoint that Bott-Samelson bimodules are very natural objects. Thus we would like a more natural geometric realization of them.

For each simple reflection s , let P_s be the minimal parabolic containing s . For a sequence $\mathbf{i} = (s, t, \dots, u)$ of simple reflections, let

$$G_{\mathbf{i}} = P_s \times_B P_t \cdots \times_B P_u.$$

We call this the **Bott-Samelson variety** corresponding to \mathbf{i} . Note that this variety still carries a $B \times B$ -action, and thus a diagonal B -action.

Furthermore, we have a projective $B \times B$ -equivariant map $m_{\mathbf{i}} : G_{\mathbf{i}} \rightarrow G$ given by multiplication, intertwining the diagonal B -action on $G_{\mathbf{i}}$ with the conjugation B -action on G .

The quotient of $G_{\mathbf{i}}$ by the right Borel action is the familiar projective Bott-Samelson variety which is used to construct resolutions of singularities for Schubert varieties. It is worth noting that just like in the flag variety case, if \mathbf{i} is a reduced expression (i.e. if $\ell(st \dots u)$ is the length of \mathbf{i}), then the multiplication map is a resolution of singularities.

Let us explain how to calculate the $B \times B$ -equivariant cohomology of the Bott-Samelson varieties. Actually, we will calculate the corresponding dg-module over $S \otimes S$. We start with a lemma:

Lemma 5.1 *Suppose $\mathcal{F} \in D_{B \times B}(pt)$ and let s be a simple reflection. Then:*

$$\Gamma_{B \times B}(\text{res}_{P_s \times B}^{B \times B} \text{ind}_{B \times B}^{P_s \times B} \mathcal{F}) = S \otimes_{S^s} \Gamma_{B \times B}(\mathcal{F})$$

Proof Thanks to Theorem 3.3 we know that $\Gamma_{P_s \times B}(\text{ind}_{B \times B}^{P_s \times B} \mathcal{F})$ is equal to $\Gamma_{B \times B}(\mathcal{F})$ regarded as an $S^s \otimes S$ -module. Hence

$$\Gamma_{B \times B}(\text{res}_{P_s \times B}^{B \times B} \text{ind}_{B \times B}^{P_s \times B} \mathcal{F}) = (S \otimes S) \overset{L}{\otimes}_{S^s \otimes S} \Gamma_{B \times B}(\mathcal{F}).$$

However, $S \otimes S$ is free as a module over $S^s \otimes S$ and is hence \mathcal{K} -projective. Thus the derived tensor product coincides with the naive tensor product and the result follows. \square

We can now prove the crucial “formality” claim mentioned above:

Proposition 5.2 *The direct images of the sheaves $\underline{\mathbb{C}}_{G_i}$ and $\mathbf{IC}(G_w)$ in $D_{B \times B}(pt)$ are formal.*

Proof First notice that we can write the sheaf $(m_i)_* \underline{\mathbb{C}}_{G_i}$ as a iterated induction and restriction:

$$(m_i)_* \underline{\mathbb{C}}_{G_i} \cong \text{res}_{P_s \times B}^{B \times B} \text{ind}_{B \times B}^{P_s \times B} \dots \text{res}_{P_t \times B}^{B \times B} \text{ind}_{B \times B}^{P_t \times B} (\underline{\mathbb{C}}_B)$$

Hence by the above lemma, letting p be the projection to a point:

$$\Gamma_{B \times B}(p_* \underline{\mathbb{C}}_{G_i}) = S \otimes_{S^s} S \otimes_{S^t} \dots \otimes_{S^u} S \quad \text{in } D_{\mathcal{A}_{B \times B}}$$

Thus the proposition is true for $\underline{\mathbb{C}}_{G_i}$. Now, by the decomposition theorem of [BBD] or more precisely, its equivariant version in [BL], we may obtain $\mathbf{IC}(G_w)$ as a direct summand of $(m_i)_* \underline{\mathbb{C}}_{G_i}$, if $\mathbf{i} = (s, \dots, s)$ is a reduced expression for w . Thus $p_* \mathbf{IC}(G_w)$ is a direct summand of $p_* \underline{\mathbb{C}}_{G_i}$ and is also formal. \square

6 Equivariant formality

Now, we will carry out some actual computations of B -equivariant cohomology, and thus of Hochschild homology.

Of course, the best setting in which to compute equivariant cohomology of a variety is when that variety (or more precisely, the sheaf one intends to compute the hypercohomology of) is equivariantly formal.

Definition/Theorem 6.1 *We call $\mathcal{F} \in D_T^b(X)$ on a T -variety X equivariantly formal if one of the following equivalent conditions holds:*

- (1) *The S module $\mathbb{H}_T^*(X, \mathcal{F})$ is free.*
- (2) *The differentials in the spectral sequence*

$$\mathbb{H}^*(X, \mathcal{F}) \otimes S \Rightarrow \mathbb{H}_T^*(X, \mathcal{F})$$

are trivial, that is, if $\mathbb{H}^(X, \mathcal{F}) \otimes S \cong \mathbb{H}_T^*(X, \mathcal{F})$ as S -modules.*

- (3) *We have the equality $\dim_{\mathbb{C}} H^*(X) = \dim_{\mathbb{C}} H^*(X^T)$.*

The equivariant formality of the Bott-Samelsons of $\mathrm{SL}(n)$ has been proven by Rasmussen in different language.

Proposition 6.2 *If $G = \mathrm{SL}(n)$ or $\mathrm{GL}(n)$, the T -space $G_{\mathbf{i}}$ is equivariantly formal for all \mathbf{i} .*

Proof By Theorem 1.2, $HH_*(R_{\mathbf{i}})$ is free as an S -module if and only if $H_T^*(G_{\mathbf{i}})$ is. By [Ra, Proposition 4.6], the module $HH_*(R_{\mathbf{i}})$ is free in type A , so by Definition/Theorem 6.1 above $G_{\mathbf{i}}$ is equivariantly formal. \square

This in turn implies that $m_*\mathbb{C}_{G_{\mathbf{i}}}$ is equivariantly formal, where $m : G_{\mathbf{i}} \rightarrow G$ is the multiplication map. Since all summands of equivariant formal sheaves are themselves equivariantly formal, and each $\mathbf{IC}(G_w)$ appears as a summand of such a sheaf (if, for example, \mathbf{i} is a reduced word for w), this completes the proof of Theorem 1.3 and the first part of Theorem 1.6.

While the most obvious consequence of equivariant formality, calculating the equivariant cohomology from ordinary or vice versa, is a rather useful one, it has less obvious ones as well.

Proposition 6.3 (Goresky, Kottwitz, MacPherson [GKM, Theorem 6.3]) *If \mathcal{F} is equivariantly formal, then the pullback map*

$$i_T^* : H_T^*(X, \mathcal{F}) \rightarrow H_T^*(X^T, \mathcal{F})$$

is injective.

As we mentioned earlier, we are interested in the Hochschild homology of Soergel bimodules as a bigraded object (so that we get a triply-graded knot homology theory), but the grading on equivariant hypercohomology is only one of these. From now on, we consider $H_T^*(G_w)$ as a bigraded S -module, with the bigrading defined by the isomorphism with Hochschild homology given by Proposition 4.1.

Proof of Theorems 1.4 and 1.6 Since the pullback map $H_T^*(G_{\mathbf{i}}) \rightarrow H_T^*(G_{\mathbf{i}}^T)$ is induced by a map of Soergel bimodules, it is homogeneous in both gradings. Similarly with the map induced by the inclusion of a summand $\mathbf{IC}(G_w) \hookrightarrow m_*\mathbb{C}_{G_{\mathbf{i}}}$. Thus we need only establish the theorem for $G_{\mathbf{i}}^T$. As this is a union of complex tori with the trivial action, we need only establish the theorem for T .

This case follows immediately from applying HH_0 to the Koszul resolution of $H_{T \times T}(T) \cong S$ as a bimodule over itself. \square

Let us turn to the case where G_w is smooth. Since $H_T^*(G_w) \cong S \otimes H^*(G_w)$, it will prove very interesting to understand $H^*(G_w)$. Surprisingly, no description of this cohomology seems to be in the literature, but in fact there is a very beautiful one.

As is well known (and we reprove in the course of Lemma 6.6 below), there exists a unique decreasing sequence of positive integers k_1, \dots, k_n such that the Hilbert series of $H^*(G_w/B)$ is of the form

$$\sum_{i=1}^{\ell(w)} q^{i/2} \dim H^i(G_w/B) = \prod_{j=1}^n \frac{1 - q^{k_j}}{1 - q}$$

Theorem 6.4 *If G_w is smooth, then as an algebra,*

$$H^*(G_w) \cong \wedge^\bullet(\gamma_1, \dots, \gamma_n) \quad (1)$$

where $\deg(\gamma_i) = 2k_i - 1$, and as an S -algebra,

$$H_T^*(G_w) \cong HH_i(R_w) \cong S \otimes H^*(G_w).$$

In the standard double grading on $HH_i(R_w)$, we have $\deg(1 \otimes \gamma_i) = (1, 2k_i)$.

This explicitly describes the Hilbert series of $HH_i(R_w)$, proving a conjecture of Rasmussen.

Corollary 6.5 *The Hilbert series of $HH_i(R_w)$ is given by*

$$\sum_{i,j} a^i q^j HH_i(R_w)_{2j} = \prod_{\ell=1}^n \frac{1 + a q^{k_\ell}}{1 - q}$$

Since $\dim H^*(G_w) = 2^{\text{rk} G}$, Definition/Theorem 6.1(3) establishes the equivariant formality of G_w independently of the earlier results of this paper (and for all types).

As usual in Lie theory, we define the height $h(\alpha) = \langle \rho^\vee, \alpha \rangle$ of a root α to be its evaluation against the fundamental coweight.

Lemma 6.6 *The cohomology ring $H^*(G_w/B)$ is a quotient of the polynomial ring S by a regular sequence (f_1, \dots, f_n) . Define $k_i = \deg(f_i)$.*

The number of times the integer m appears in the list k_1, \dots, k_n is precisely the number of roots in $R^+ \cap w^{-1}(R^-)$ of height $m - 1$ minus the number of such roots of height m , where R^+ is the set of positive roots of G and $R^- = -R^+$.

Proof For ease, assume G_w is not contained in a parabolic subgroup. By results of Akyıldız and Carrell [AC], the ring $H^*(G_w/B)$ is a quotient of $\mathbb{C}[BwB/B]$ by a regular sequence of length $\ell(w)$ by a regular sequence (g_1, \dots, g_k) where $k = \ell(w)$.

In this grading, $\mathbb{C}[BwB/B]$ is a polynomial algebra generated by elements of degree $2h(\alpha)$ for each root $\alpha \in R^+ \cap w^{-1}(R_-)$, and the degrees of g_i are given by $2h(\alpha) + 2$ as α ranges over the same roots. Corresponding to the simple roots are $n - 1$ generators x_1, \dots, x_n of degree 2, which are the first Chern classes of line bundles on G/B corresponding the fundamental weights, and for the other roots we have $\ell(w) - n + 1$ other generators $y_n, \dots, y_{\ell(w)}$ of higher degree. Here, we assume these are in increasing order by degree.

It is a well known fact that the cohomology ring $H^*(G/B)$ is generated by the Chern classes x_i . Since the natural pullback map $H^*(G/B) \rightarrow H^*(G_w/B)$ is onto, the ring $H^*(G_w/B)$ is as well. That is, if $p = \deg y_k > 2$, then

$$y_k - \sum_{\deg(g_i)=p} \beta_i g_i \in \sum_{\deg(g_j)<p} Sg_j.$$

Since $y_k \notin \sum_{\deg(g_j)<p} Sg_j$, we can eliminate y_k and any single relation g_i such that $\beta_i \neq 0$. Obviously, $(g_1, \dots, g_n) \setminus \{g_i\}$ is again a regular sequence in $\mathbb{C}[x_1, \dots, x_n, y_n, \dots, y_{k-1}]$. Applying this argument inductively, we obtain a subsequence $(g_{i_1}, \dots, g_{i_n})$ which is regular in S , which is the desired sequence.

Since the number of relations of degree j which have been eliminated is the number of generators of degree j , the remaining number of relations is precisely the difference between these, which is also the number of roots of height $j/2 - 1$ minus the number of height $j/2$ in $R^+ \cap w^{-1}(R^-)$. \square

Proof of Theorem 6.4 Applying the Hirsch lemma (as stated in the paper [DGMS]) to the fibration $B \rightarrow G_w \rightarrow G_w/B$, we see that the cohomology ring $H^*(G_w)$ is the cohomology of the dg-algebra $H^*(G_w/B) \otimes_S \mathcal{K}_T$ where \mathcal{K}_T is the Koszul complex of S (the natural free resolution of \mathbb{C} as an S -module equipped with the Yoneda product).

Since (f_1, \dots, f_n) is regular, $H^*(G_w/B)$ is quasi-isomorphic to the Koszul complex \mathcal{K}_f . Thus, the cochain complex of G_w is quasi-isomorphic to $\mathcal{K}_f \otimes_S \mathbb{C}$. This is just an exterior algebra over \mathbb{C} with generators $\gamma_1, \dots, \gamma_n$ with degree given by $\deg(\gamma_i) = 2k_i - 1$.

Since we used a Koszul complex, the generators γ land in Hochschild degree 1 under the restriction map to T , so the degree of γ_i is $(2k_i, 1)$.

If $w \in W'$, where W' is a proper parabolic subgroup of W (with corresponding Levi subgroup $G' \subset G$), and W' is the minimal such parabolic. Then $G_w \cong G'_w \times T/T' \times N \cap w'_0 N w'_0$. Since $N \cap w'_0 N w'_0$ is unipotent and thus contractible, by Künneth, we have

$$\dim_{\mathbb{C}} H^*(G_w) \cong \dim_{\mathbb{C}} H^*(G_{w'}) \dim_{\mathbb{C}} H^*(T/T') = 2^{\dim T'} 2^{\dim T/T'} = 2^{\dim T}.$$

Thus, G_w is equivariantly formal and the degrees k_i for G_w are simply those of $G_{w'}$ extended by adding 1's. \square

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